$*$-SDYM fields and heavenly spaces: II. Reductions of the $*$-SDYM system.

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Abstract. Reductions of self-dual Yang-Mills (SDYM) system for $*$-bracket Lie algebra to the Husain-Park (HP) heavenly equation and to $sl(N, \boldsymbol{C})$ SDYM equation are given. An example of a sequence of $su(N)$ chiral fields ($N \geq 2$) tending for $N \to \infty$ to a curved heavenly space is found.

Keywords: integrable systems, self-dual gravity, Fedosov $*$-product.

Introduction

This paper is a second part of our previous work (Formański and Przanowski 2005) where the integrability of self-dual Yang-Mills (SDYM) system in the formal $*$-algebra bundle over heavenly space has been proved. In what follows we refer to the first paper as I. In particular it has been shown in I, that the considered SDYM equations can be reduced to one equation called *master equation* (ME)

$$g^{\tilde{\beta}\alpha}\partial_{\tilde{\beta}}\partial_\alpha\Theta + \frac{1}{2G}\epsilon^{\alpha\beta}\{\partial_\alpha\Theta\, ,\, \partial_\beta\Theta\} = 0 \tag{1}$$

Notation in the present paper is the same as in the first one. So $g_{\alpha\tilde{\beta}}$ are components of the Kăhler metric on the base manifold $\mathcal{M}$ i.e. $g_{\alpha\tilde{\beta}} = \partial_\alpha\partial_{\tilde{\beta}}\mathcal{K}$, where $\mathcal{K}$ is a Kăhler potential. As $\mathcal{M}$ is a four dimensional heavenly space $\det(g_{\alpha\tilde{\beta}}) = G(w,z)\tilde{G}(\tilde{w},\tilde{z})$, with $\{w,\tilde{w},z,\tilde{z}\}$ being local coordinates on $\mathcal{M}$. The function Θ takes values in the formal $*$-algebra $(\mathcal{A},*)$ of formal power series over a symplectic manifold $(\Sigma^{2n},\boldsymbol{\omega})$

$$\Theta = \sum_{m=0}^{\infty}\sum_{k=-m}^{\infty} t^m\hbar^k\Theta_{m,k}(w,z,\tilde{w},\tilde{z},x^1,...,x^{2n}) \tag{2}$$

where $\{x^1,...,x^{2n}\}$ are local coordinates on Σ^{2n} and $\Theta_{m,k}(w,z,\tilde{w},\tilde{z},x^1,...,x^{2n})$ are analytic functions on $\mathcal{M}\times\Sigma^{2n}$. The $(\mathcal{A},*)$ algebra has been discussed in detail in I.

The evidence of integrability of ME has been achieved by:
• construction of infinite hierarchy of conservation laws,

• rewritting the equation (1) as an integrability condition for a Lax pair,
• twistor construction i.e. one-to-one correspondence between solution of (1) and vector bundles over twistor space for $\mathcal{M}$.
• the proof of existence of a solution for a respective Riemann-Hilbert problem.

In the present work we study the role which the equation (1) plays among other integrable systems. As it was discussed previously (cf also Mason and Woodhouse 1996) the SDYM system for finite diemensional Lie algebras is not general enough to include all lower diemensional integrable systems (see also (Ward 1992)). In order to generalize the SDYM system to obtain a universal one according to the famous *Ward's conjecture* (Ward 1985) it seems natural to extend the algebra of this system to the one containing all $sl(N,\boldsymbol{C})$ algebras as well as infinite diemensional algebra of Hamiltonian vector fields over $(\Sigma^{2n},\boldsymbol{\omega})$. In this paper we show that the algebra chosen in (Formański and Przanowski 2005) meets this requirements. We find a reduction of ME to the $sl(N,\boldsymbol{C})$ SDYM equation for $N\geq 2$ as well as to a heavenly equation. This also opens the possibiliy to find a sequence of $su(N)$ chiral fields tending to a heavenly space as $N\to\infty$. The last problem was first posed by Ward (1990b).

The paper is organized as follows. Section 1 is devoted to the reduction of ME to the Husain-Park (HP) heavenly equation. We concentrate mainly on solutions of ME which are analytic functions of deformation parameters $\hbar$ and t. The example of such a solution is given. The heavenly space obtained is analysed in terms of the null tetrad formalism.

In Section 2 we consider reduction of ME to the $sl(N,\boldsymbol{C})$ SDYM equation. We describe a basis of $sl(N,\boldsymbol{C})$ introduced by (Farlie et al 1990). Theorem 2.1, which is a main result of this section proves that any analytic solution of the $sl(N,\boldsymbol{C})$ SDYM equation can be obtained by a reduction of some solution to ME.

Section 3 deals with the relations between $sl(N,\boldsymbol{C})$ chiral equation and ME. We propose a precise meaning of limitting process $N\to\infty$ for $sl(N,\boldsymbol{C})$ chiral fields. An example of $su(N)$ chiral field sequence tending to a heavenly space is given.

Concluding remarks close the paper.

1 Reduction of ME to the heavenly equation

The form of ME (1) is quite general. This is due to the fact that it is given over an arbitrary heavenly space $\mathcal{M}$ and the algebra in which the function Θ takes values is an arbitrary formal $*$-algebra $(\mathcal{A},*)$ over any symplectic manifold $(\Sigma^{2n},\boldsymbol{\omega})$.

In this section we restrict ourselves to the base manifold $\mathcal{M}=\boldsymbol{C}^4$ (or $\boldsymbol{R}^4$ of the signature $(++--)$) with Kǎhler potential $\mathcal{K}=w\tilde{w}+z\tilde{z}$. Also the symplectic manifold is assumed to be two dimensional. Thus ME (1) takes the form

$$\partial_w\partial_{\tilde{w}}\Theta+\partial_z\partial_{\tilde{z}}\Theta+\{\partial_w\Theta\,,\,\partial_z\Theta\}=0. \tag{1.1}$$

We seek for the solutions of the equation (1.1) which are analytic in all coordinates and also in both parameters t and $\hbar$. It means that the series (2) is convergent. Then we can interchange the sums i.e. rewrite the function Θ in the form

$$\Theta = \sum_{k=0}^{\infty} \hbar^k \Theta_k(t, w, z, \tilde{w}, \tilde{z}, x^1, x^2) \tag{1.2}$$

where each $\Theta_k(t, w, z, \tilde{w}, \tilde{z}, x^1, x^2)$ is analytic in all varaibles $\{t, w, z, \tilde{w}, \tilde{z}, x^1, x^2\}$.

Note, that this case explains why the parameter t has been called in I the *convergence parameter*. Observe also that in many such cases the parameter t can be absorbed by a suitable redefinition of coordinates $\{w, z, \tilde{w}, \tilde{z}\}$ on $\mathcal{M}$.

As is known the $*$-product has a form of formal power series

$$\forall f, g \in C^{\omega}(\Sigma^{2n}) \qquad f * g = \sum_{k=0}^{\infty} \hbar^k \Delta_k(f, g) \tag{1.3}$$

The analyticity of $\Theta(t, \hbar; w, z, \tilde{w}, \tilde{z}, x^1, x^2)$ in t and $\hbar$ imposes strong requirements on the $*$-product itself. This product should be also analytic function of the deformation parameter. Thus the series (1.3) should be convergent at least for some neighbourhood of $\hbar = 0$.

To reduce two dimensions in (1.1) we impose the symmetry in two orthogonal directions $(\partial_w - \partial_{\tilde{w}})\Theta = 0$ and $(\partial_z - \partial_{\tilde{z}})\Theta = 0$. This leads to

$$\partial_w^2 \Theta + \partial_z^2 \Theta + \{\partial_w \Theta \, , \, \partial_z \Theta\} = 0. \tag{1.4}$$

for $\Theta = \Theta(t, \hbar, w + \tilde{w}, z + \tilde{z}, x^1, x^2)$.

Substitute $w + \tilde{w} \mapsto w$ and $z + \tilde{z} \mapsto z$ and define $\theta := \Theta_0(t, w, z, x^1, x^2)$. From (1.2) one gets $\theta = \lim_{\hbar \to 0} \Theta$. Then, from the definition of deformation quantization, the bracket $\{\partial_w \Theta, \partial_z \Theta\}$ turns into Poisson bracket $\{\partial_w \theta, \partial_z \theta\}_{\text{Poisson}}$ in the classical limit $\hbar \to 0$. In this limit the equation (1.4) leads to the *Husain-Park* (HP) *heavenly equation* (Park 1992, Husain 1994)

$$\partial_w^2 \theta + \partial_z^2 \theta + \{\partial_w \theta \, , \, \partial_z \theta\}_{\text{Poisson}} = 0 \tag{1.5}$$

(Comparing ME (1.1) with (1.5) we conclude that ME can be thought of as a quantum deformation of HP heavenly equation lifted to six dimensions.)

The solution $\theta(t, w, z, x^1, x^2)$ of (1.5) defines the heavenly metric

$$\begin{aligned} ds^2 \quad = \quad & dw \cdot (\partial_{x^1} \partial_w \theta dx^1 + \partial_{x^2} \partial_w \theta dx^2) + dz \cdot (\partial_{x^1} \partial_z \theta dx^1 + \partial_{x^2} \partial_z \theta dx^2) + \\ & - \frac{1}{\{\partial_w \theta, \partial_z \theta\}_{\text{Poisson}}} [(\partial_{x^1} \partial_w \theta dx^1 + \partial_{x^2} \partial_w \theta dx^2)^2 + (\partial_{x^1} \partial_z \theta dx^1 + \partial_{x^2} \partial_z \theta dx^2)^2]. \end{aligned} \tag{1.6}$$

Remark 1

- On the other hand one can consider the (formal) deformation quantization of (1.5). This leads to the equation (1.4) but written in the algebra of formal power series i.e.

$$\Theta = \sum_{k=0}^{\infty} \hbar^k \Theta_k(w, z, x^1, x^2)$$

This algebra is in general different from the algebra $\mathcal{A}$ and from the analytic algebra considered above. The series may not be convergent, but the reduction to the HP equation exists. This reduction corresponds to the classical limit and in this limit any $*$-product and any algebra of quantum observables reduce to the algebra of functions with Poisson bracket.

Example 1 In order to find a solution of (1.4) consider the Cauchy data

$$\begin{aligned} \Theta|_{z=0} &= \frac{\pi}{2}\cos(x^1+x^2) - w\sin x^2 \\ \partial_z\Theta|_{z=0} &= -\sin x^1 \end{aligned}$$

One easily computes from (1.4) that for $k \geq 2$

$$\begin{aligned} \tfrac{\partial^k\Theta}{\partial z^k}|_{z\,=\,0} &= \{...\{\partial_z\Theta, \underbrace{\partial_w\Theta\}, ..., \partial_w\Theta\}}_{k\,-\,1\text{ times}}|_{z\,=\,0} \\ &= -(\tfrac{2}{\hbar}\sin\tfrac{\hbar}{2})^{k-1}\cos^{k-1}x^2\,\tfrac{d^{k-2}}{d(x^1)^{k-2}}\cos x^1 \end{aligned}$$

The Cauchy-Kowalewska form of the solution $\Theta = \sum \frac{1}{k!}\frac{\partial^k\Theta}{\partial z^k}|_{z\,=\,0}z^k$ reads

$$\begin{aligned} \Theta &= \Theta|_{z=0} + \partial_z\Theta|_{z=0}z + \sum_{m=1}^{\infty}\frac{(-1)^m}{2m!}(\frac{2}{\hbar}\sin\frac{\hbar}{2}\cos x^2)^{2m-1}(\cos x^1)\,z^{2m} \\ &+ \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m+1)!}(\frac{2}{\hbar}\sin\frac{\hbar}{2}\cos x^2)^{2m}(\sin x^1)\,z^{2m+1} \end{aligned} \tag{1.7}$$

Finally, the solution of (1.4) is given by

$$\Theta = \frac{\pi}{2}\cos(x^1+x^2) - w\sin x^2 + \frac{\cos[z\cdot\frac{2}{\hbar}\sin\frac{\hbar}{2}\cdot\cos x^2 + x^1] - \cos x^1}{\frac{2}{\hbar}\sin\frac{\hbar}{2}\cos x^2} \tag{1.8}$$

This is an analytic function of $\hbar$. The absence of the parameter t may be thought of as a consequence of a suitable choice of coordinates w and z.

To simplify the notation we denote $p \equiv x^1$ and $q \equiv x^2$. Thus the solution $\Theta = \Theta(\hbar, w, z, p, q)$ and (1.8) reads

$$\begin{aligned} \Theta &= \frac{\pi}{2}\cos(p+q) - w\sin q + \frac{\cos[z\cdot\frac{2}{\hbar}\sin\frac{\hbar}{2}\cdot\cos q + p] - \cos p}{\frac{2}{\hbar}\sin\frac{\hbar}{2}\cos q} \\ &= \frac{\pi}{2}\cos(p+q) - w\sin q - \int_0^z \sin(\zeta\frac{2}{\hbar}\sin\frac{\hbar}{2}\cdot\cos q + p)d\zeta \end{aligned}$$

In the limit $\hbar \to 0$ one gets the solution of (1.5)

$$\begin{aligned} \theta &= \frac{\pi}{2}\cos(p+q) - w\sin q + \frac{\cos[z\cdot\cos q + p] - \cos p}{\cos q} \\ &= \frac{\pi}{2}\cos(p+q) - w\sin q - \int_0^z \sin(\zeta\cdot\cos q + p)d\zeta \end{aligned}$$

Substituting this θ into (1.6) we obtain the following heavenly metric

$$\begin{aligned} ds^2 &= -\cos q\, dw\, dq + \cos(z\cos q + p)\cdot(z\sin q\, dq - dp)dz \qquad (1.9)\\ &\quad -\frac{1}{\cos q\cos(z\cos q+p)}[\cos^2 q\, dq^2 + \cos^2(z\cos q+p)\cdot(z\sin q\, dq - dp)^2] \end{aligned}$$

In order to classify the heavenly metric (1.9), we rewrite it in the null tetrad form $ds^2 = 2\boldsymbol{e}^1\boldsymbol{e}^2 + 2\boldsymbol{e}^3\boldsymbol{e}^4$. In what follows we use the convention of Plebański (1975), Plebański and Przanowski (1988). One quickly finds that the 1-forms

$$\begin{aligned} \boldsymbol{e}^1 &= \frac{1}{\sqrt{2}}\Phi^{-1}[\cos q\, dz - (z\sin q\, dq - dp)]\\ \boldsymbol{e}^2 &= \frac{1}{\sqrt{2}}(z\sin q\, dq - dp)\\ \boldsymbol{e}^3 &= -\frac{1}{\sqrt{2}}dq\\ \boldsymbol{e}^4 &= \frac{1}{\sqrt{2}}[\cos q\, dw + \Phi\, dq] \end{aligned}$$

where $\Phi := \frac{\cos q}{\cos(z\cos q+p)}$, define a null tetrad for the metric (1.9). Then

$$\begin{aligned} dq &= -\sqrt{2}\boldsymbol{e}^3 \quad &,\quad dw &= \tfrac{\sqrt{2}}{\cos q}(\boldsymbol{e}^4 + \Phi\,\boldsymbol{e}^3)\\ dp &= -\sqrt{2}(\boldsymbol{e}^2 + z\sin q\boldsymbol{e}^3) \quad &,\quad dz &= \tfrac{\sqrt{2}}{\cos q}(\boldsymbol{e}^2 + \Phi\,\boldsymbol{e}^1) \end{aligned}$$

The first Cartan structure equations $d\boldsymbol{e}^a = -\Gamma^a{}_b \wedge \boldsymbol{e}^b$ read

$$\begin{aligned} d\boldsymbol{e}^1 &= -\sqrt{2}\,\mathrm{tg}\, q\, \boldsymbol{e}^3 \wedge \boldsymbol{e}^1\\ d\boldsymbol{e}^2 &= -\sqrt{2}\,\mathrm{tg}\, q\,[\boldsymbol{e}^2 \wedge \boldsymbol{e}^3 + \Phi\,\boldsymbol{e}^1 \wedge \boldsymbol{e}^3]\\ d\boldsymbol{e}^3 &= 0\\ d\boldsymbol{e}^4 &= \sqrt{2}\,[\,\mathrm{tg}\, q\boldsymbol{e}^3 \wedge \boldsymbol{e}^4 + \Phi^2\mathrm{tg}\,(z\cos q + p)\boldsymbol{e}^3 \wedge \boldsymbol{e}^1\,] \end{aligned}$$

Consequently, the only non zero connection 1-forms are

$$\begin{aligned} \Gamma_{12} &= -\sqrt{2}\mathrm{tg}\, q\, \boldsymbol{e}^3\\ \Gamma_{31} &= -\sqrt{2}\Phi\,[\mathrm{tg}\, q\, \boldsymbol{e}^1 \;+\; \Phi\,\mathrm{tg}\,(z\cos q + p)\,\boldsymbol{e}^3]\\ \Gamma_{34} &= -\sqrt{2}\mathrm{tg}\, q\, \boldsymbol{e}^3 \end{aligned}$$

First note that the *dotted* spinor connection $\Gamma_{\dot{A}\dot{B}}$ vanishes (Plebański 1975, Plebański and Przanowski 1988) since

$$\Gamma_{41} = \frac{1}{2}(-\Gamma_{12} + \Gamma_{34}) = \Gamma_{32} = 0 \quad \Rightarrow \quad \Gamma_{\dot{A}\dot{B}} = 0.$$

Thus the curvature form is self-dual and indeed, the metric ds^2 given by (1.9) describes the heavenly space.

On the other hand the only nontrivial second Cartan structure equation for the *undotted spinor connection* is

$$d\Gamma_{31} + (\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{31} = \frac{1}{2} C^{(1)} \boldsymbol{e}^3 \wedge \boldsymbol{e}^1.$$

where $\frac{1}{2} C^{(1)} = C_{2222}$ and C_{ABCD} is *undotted Weyl spinor* i.e. the spinor image of the self-dual part of the Weyl tensor.

This gives

$$C^{(1)} = 4\Phi \, [\frac{1 + 2\sin^2 q}{\cos^2 q} \; + \; \frac{1 + 2\sin^2 (z\cos q + p)}{\cos^2 (z\cos q + p)} \Phi^2 \,].$$

Thus the only nonvanishing component of the Weyl spinor is $C^{(1)}$, what means that the heavenly space $\mathcal{M}$ is of the type $[4] \times [-]$ $(N \times 0)$.

Summarizing, the solution (1.8) leads in a classical limit $\hbar \to 0$ to a heavenly space of the type $[4] \times [-]$ described by the metric (1.9). This metric is complex if $\{w, z, p, q\}$ are complex or real of signature $(++--)$ if $\{w, z, p, q\}$ are real.

As has been pointed out by Maciej Dunajski the transformation $u = \sin q$, $v = \sin(z\cos q + p)$ brings the metric (1.9) to a simple pp-wave (Plebański 1975) form

$$ds^2 = -\, dw\, du + dv\, dz - \frac{1}{\sqrt{(1-u^2)(1-v^2)}} (du^2 + dv^2)$$

2 Reduction of ME to the $sl(N, \boldsymbol{C})$ SDYM equation

In this section we examine the conditions under which the solution of ME (1) defines an analytic solution of $sl(N, \boldsymbol{C})$ SDYM equation.

As has been pointed out in the remark 1 of the previous section, the reduction of ME to HP heavenly equation is in a sense trivial. This is due to the fact that Poisson algebra (the algebra of Hamiltonian vector fields over $(\Sigma^2, \boldsymbol{\omega})$) by the very definition can be embedded into any deformed algebra.

In the case of reduction to $sl(N, \boldsymbol{C})$ algebra one has to find an algebra $(\mathcal{A}, *)$ for which there exists a Lie algebra homomorphism $\mathcal{A} \to sl(N, \boldsymbol{C})$.

The noncommutative $*$-product exists on each symplectic space $(\Sigma^{2n}, \boldsymbol{\omega})$. This was proved by De Wilde and Lacomte (1992) and Fedosov (1994, 1996). In this paper we restrict ourselves to two dimensional torus T^2 with coordinates (x^1, x^2) and symplectic form $\boldsymbol{\omega} = dx^1 \wedge dx^2$. Fedosov's $*$ multiplication on T^2 can be chosen to be just the Moyal $*$-product (Fedosov 1994, 1996).

$$\forall f, g \in C^\omega(T^2) \qquad f * g := \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{i\hbar}{2})^k \omega^{i_1 j_1} ... \omega^{i_k j_k} \frac{\partial^k f}{\partial x^{i_1} ... \partial x^{i_k}} \frac{\partial^k g}{\partial x^{j_1} ... \partial x^{j_k}}.$$

Each analytic function f on T^2, $f \in C^\omega(T^2)$ can be expanded into Fourier series with respect to the basis

$$E_{\vec{m}} = \exp[i(m_1 x^1 + m_2 x^2)] \,, \qquad \vec{m} := (m_1, m_2) \in \boldsymbol{Z} \times \boldsymbol{Z}. \qquad (2.1)$$

In this basis the Moyal $*$-product has a simple form

$$E_{\vec{m}} * E_{\vec{n}} = \exp(\frac{i\hbar}{2}\vec{m}\times\vec{n})E_{\vec{m}+\vec{n}}\,; \qquad \vec{m}\times\vec{n} = m_1n_2 - m_2n_1. \tag{2.2}$$

Consequently the Moyal bracket

$$\{E_{\vec{m}}, E_{\vec{n}}\} = \frac{2}{\hbar}\sin(\frac{\hbar}{2}\vec{m}\times\vec{n})E_{\vec{m}+\vec{n}}. \tag{2.3}$$

Remark 2

- The formulas (2.2) and (2.3) prove that the Moyal $*$-product and Moyal bracket are analytic in deformation parameter $\hbar$.

  This feature is fundamental to find a reduction of the formal $*$-algebra to a finite dimensional algebra. It makes possible to put a particular value of $\hbar$ and then to come from formal power series to analytic functions. Then in this case one can expect that the solutions of the $*$-SDYM equation analytic in $\hbar$ define a family of solutions to $sl(N,\boldsymbol{C})$ SDYM equations.

In order to find a relation between the Lie algebras $(C^{\omega}(T^2), \{\cdot,\cdot\})$ and $sl(N,\boldsymbol{C})$ we have to introduce a suitable basis for this latter algebra.

2.1 Trygonometric structure constants for $sl(N,\boldsymbol{C})$

Here we briefly summarize the results of (Fairlie et al 1990). In that distinguished work the basis for finite diemensional Lie algebras was constructed in which the structure constants are trygonometric functions.

For any $\vec{m} := (m_1, m_2) \in \boldsymbol{Z}\times\boldsymbol{Z}$, where $\boldsymbol{Z}$ is a group of rational numbers, consider an $N\times N$ matrix

$$L_{\vec{m}} := \frac{iN}{2\pi}\omega^{\frac{m_1m_2}{2}}S^{m_1}T^{m_2}. \tag{2.4}$$

where

$$S := \sqrt{\omega}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}, \quad T := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \\ -1 & 0 & .0 & \dots & 0 \end{pmatrix}$$

and ω is the Nth root of 1. We take $\omega := exp(\frac{2\pi i}{N})$.

The unitary matrices S and T satisfy: $T\,S = \omega S\,T$, and $S^N = T^N = -\mathbf{1}$.

The system of (N^2-1) matrices $\{L_{\vec{\mu}}\}$, $0 \le \mu_1 < N$, $0 \le \mu_2 < N$ apart from $(\mu_1,\mu_2) = (0,0)$, is linearly independent.

Proposition 2.1 *Matrices $L_{\vec{m}}$ have the following properties*

1. $L_{\vec{m}+N\vec{r}} = (-1)^{(m_1+1)r_2+(m_2+1)r_1+Nr_1r_2}L_{\vec{m}}$

2. $\mathrm{Tr} L_{\vec{m}} = 0$ apart from $m_1 = m_2 = 0$ modulo N

3. $TrL_{N\vec{r}} := (-1)^{r_2+r_1+Nr_1r_2}\, \frac{iN^2}{2\pi}$.

4. $L_{\vec{m}} L_{\vec{n}} = \frac{iN}{2\pi}\omega^{\frac{\vec{n}\times\vec{m}}{2}} L_{\vec{m}+\vec{n}}; \quad \vec{n}\times\vec{m} := n_1m_2 - n_2m_1$

5. $L^{\dagger}_{\vec{m}} = -L_{-\vec{m}} = (\frac{N}{2\pi})^2 L^{-1}_{\vec{m}}$

6. $\det L_{\vec{m}} := (-1)^{N(m_1+m_2+m_1m_2)}\, (\frac{iN}{2\pi})^N$ $\Box$

Corollary 2.2

1. The matrices $L_{\vec{\mu}}$, $0 \leq \mu_1 < N$, $0 \leq \mu_2 < N$ and $\vec{\mu} \neq (0,0)$ are traceless and linearly independent. Therefore, they constitute a basis of the $sl(N, \boldsymbol{C})$ algebra. For this basis the structure constants are trygonometric functions

$$[L_{\vec{\mu}}, L_{\vec{\nu}}] = \frac{N}{\pi} sin(\frac{\pi}{N}\vec{\mu}\times\vec{\nu}) L_{\vec{\mu}+\vec{\nu}} \tag{2.5}$$

2. Appropriate linear combinations of $L_{\vec{\mu}}$ define anti-hermitian matrices which constitute a basis for $su(N)$ algebra, for arbitrary $N \geq 2$. $\Box$

Remark 3

- In all the paper the Greek indecies $\vec{\mu}, \vec{\nu}, \ldots$ etc. satisfy $\vec{\mu} := (\mu_1, \mu_2) \neq (0,0)$ and $0 \leq \mu_1 \leq N-1$, $0 \leq \mu_2 \leq N-1$.

2.2 From the solution of ME to the solution of the $sl(N, \boldsymbol{C})$ SDYM equation

As has been said at the beginning of this section, the Moyal bracket $\{E_{\vec{m}}, E_{\vec{n}}\} = \frac{2}{\hbar}\sin(\frac{\hbar}{2}\vec{m}\times\vec{n})E_{\vec{m}+\vec{n}}$ is an analytic function of the deformation parameter. Thus one can replace the formal power series by convergent ones and put a particular value for $\hbar$. Taking $\hbar = \frac{2\pi}{N}$ one gets

$$\{E_{\vec{m}}, E_{\vec{n}}\} = \frac{N}{\pi}\sin(\frac{\pi}{N}\vec{m}\times\vec{n})E_{\vec{m}+\vec{n}}. \tag{2.6}$$

The structure constants are the same as the structure constants of $sl(N, \boldsymbol{C})$ algebra in the basis $L_{\vec{\mu}}$. For this reason, for each $N \geq 2$, we can define a Lie algebra homomorphism $\chi_N : Span\{E_{\vec{m}}\} \xrightarrow{\text{onto}} sl(N, \boldsymbol{C})$, by

$$\chi_N : \begin{cases} E_{\vec{\mu}+N\vec{r}} & \longmapsto \quad (-1)^{(\mu_1+1)r_2+(\mu_2+1)r_1+Nr_1r_2} L_{\vec{\mu}} \\ E_{N\vec{r}} & \longmapsto \quad 0 \end{cases} \tag{2.7}$$

where $\vec{\mu}$ has been defined in the remark 3 and $\vec{r} := (r_1, r_2) \in \boldsymbol{Z}\times\boldsymbol{Z}$.

Recall, that in the K-Newman formalism the $sl(N, \boldsymbol{C})$ SDYM system on the heavenly background can be reduced to one equation

$$g^{\tilde{\beta}\alpha}\partial_\alpha\partial_{\tilde{\beta}}\vartheta + \frac{1}{2G}\epsilon^{\alpha\beta}[\partial_\alpha\vartheta\,,\,\partial_\beta\vartheta] = 0\,; \qquad \vartheta(y,\tilde{y},z,\tilde{z}) \in sl(N, \boldsymbol{C}) \tag{2.8}$$

(Newman 1978, Leznov 1988, Parkes 1992, Plebański and Przanowski 1996, Mason and Woodhouse 1996, Przanowski and Formański 1999, Formański 2004).

The following theorem gives the relations between ME (1) and $sl(N,\boldsymbol{C})$ SDYM equation (2.8) (see also Przanowski and Formański (1999)).

Theorem 2.1 *Each analytic solution of $sl(N,\boldsymbol{C})$ SDYM equation (2.8) is an image in the homomorphism χ_N of some solution of ME (1).*

Proof. Suppose that ϑ satisfies the equation (2.8). Without any loss of generality we can take such coordinates on $\mathcal{M}$ that $g_{z\tilde{z}} \neq 0$ i.e. $g^{\tilde{w}w} \neq 0$. After the change of coordinates

$$w = \frac{1}{2}(y+\tilde{y})\,, \quad \tilde{w} = \frac{1}{2}(y-\tilde{y}) \quad \Rightarrow \quad y = w+\tilde{w}\,, \quad \tilde{y} = w-\tilde{w} \tag{2.9}$$

the $sl(N,\boldsymbol{C})$ SDYM equations (2.8) reads

$$\begin{aligned}\partial_y^2\vartheta \;\; = \;\; & \partial_{\tilde{y}}^2\vartheta - \frac{1}{g^{\tilde{w}w}}(\, g^{\tilde{z}z}\partial_z\partial_{\tilde{z}}\vartheta + g^{\tilde{z}w}\partial_y\partial_{\tilde{z}}\vartheta + g^{\tilde{z}w}\partial_{\tilde{y}}\partial_{\tilde{z}}\vartheta + g^{\tilde{w}z}\partial_z\partial_y\vartheta - g^{\tilde{w}z}\partial_z\partial_{\tilde{y}}\vartheta\,) \\ & - \frac{1}{G\,g^{\tilde{w}w}}\,[\partial_y\vartheta + \partial_{\tilde{y}}\vartheta\,,\,\partial_z\vartheta]\end{aligned} \tag{2.10}$$

Each analytic solution of the above equation can be obtained by the Cauchy-Kowalewska method for some Cauchy data. Assume the Cauchy data of the form

$$\begin{aligned}\vartheta|_{y=0} &= \sum_{\vec{\mu}} \vartheta^{(0)}_{\vec{\mu}}(\tilde{y},z,\tilde{z})L_{\vec{\mu}} \\ \partial_y\vartheta|_{y=0} &= \sum_{\vec{\mu}} \vartheta^{(1)}_{\vec{\mu}}(\tilde{y},z,\tilde{z})L_{\vec{\mu}}\end{aligned} \tag{2.11}$$

(Remember that the index $\vec{\mu}$ in above sums satisfies the conditions of remark 3).

Consider ME. Under the same assumptions about $\mathcal{M}$ it takes the same form as (2.10)

$$\begin{aligned}\partial_y^2\Theta \;\; = \;\; & \partial_{\tilde{y}}^2\Theta - \frac{1}{g^{\tilde{w}w}}(\, g^{\tilde{z}z}\partial_z\partial_{\tilde{z}}\Theta + g^{\tilde{z}w}\partial_y\partial_{\tilde{z}}\Theta + g^{\tilde{z}w}\partial_{\tilde{y}}\partial_{\tilde{z}}\Theta + g^{\tilde{w}z}\partial_z\partial_y\Theta - g^{\tilde{w}z}\partial_z\partial_{\tilde{y}}\Theta) \\ & - \frac{1}{G\,g^{\tilde{w}w}}\{\partial_y\Theta + \partial_{\tilde{y}}\Theta\,,\,\partial_z\Theta\}\,)\end{aligned} \tag{2.12}$$

One can seek for the solution of (2.12) for which the Cauchy data as projected by χ_N give (2.11) i.e.

$$\begin{aligned}\Theta|_{y=0} &= \sum_{\vec{\mu}} \vartheta^{(0)}_{\vec{\mu}}(\tilde{y},z,\tilde{z})E_{\vec{\mu}} \\ \partial_y\Theta|_{y=0} &= \sum_{\vec{\mu}} \vartheta^{(1)}_{\vec{\mu}}(\tilde{y},z,\tilde{z})E_{\vec{\mu}}\end{aligned} \tag{2.13}$$

Thus the Cauchy-Kowalewska method in both cases gives the solution in a form of a power series with respect to the coordinates $\{y, \tilde{y}, z, \tilde{z}\}$. For simplicity we write here

$$\vartheta = \sum_{j=0}^{\infty} \frac{1}{j!} (\sum_{\vec{\mu}} \vartheta^{(j)}_{\vec{\mu}} (\tilde{y}, z, \tilde{z})\, L_{\vec{\mu}})\, y^j \qquad \text{in} \quad sl(N, \boldsymbol{C}). \tag{2.14}$$

$$\Theta = \lim_{K\to\infty} \sum_{j=0}^{K} \frac{1}{j!} (\sum_{\vec{n}=\vec{0}}^{\vec{M}_j} \Theta^{(j)}_{\vec{n}} (\hbar; \tilde{y}, z, \tilde{z}) E_{\vec{n}}\,) y^j \qquad \text{in} \quad C^{\omega}(T^2). \tag{2.15}$$

In the last sum the symbol $\sum_{\vec{n}=\vec{0}}^{\vec{M}_j}$ means sum over $0 \leq n_1 \leq M_{1j}$, $0 \leq n_2 \leq M_{2j}$, where for each j the numbers M_{1j} and M_{2j} are finite.
At each order K of the approximation we have

$$\chi_N (\sum_{j=0}^{K} \frac{1}{j!} (\sum_{\vec{n}=\vec{0}}^{\vec{N}_j} \Theta^{(j)}_{\vec{n}} (\frac{2\pi}{N}; \tilde{y}, z, \tilde{z}) E_{\vec{n}}\, y^j\,)) = \sum_{\vec{\mu}} \sum_{j=0}^{K} (\, \frac{1}{j!} \vartheta^{(j)}_{\vec{\mu}} (\tilde{y}, z, \tilde{z})\, y^j\,) L_{\vec{\mu}}$$

Thus the solution (2.14) is defined by the solution of (2.15) and it can be called an image of the solution (2.15) in χ_N.
This completes the proof. □

The above theorem says that the SDYM equation in $sl(N, \boldsymbol{C})$ algebra for arbitrary $N \geq 2$ is a reduction of ME. As is known from the works (Mason and Sparling 1989, Chakravarty and Ablowitz 1992, Tafel 1993) the $sl(N, \boldsymbol{C})$ SDYM equations reduce to many integrable systems. Thus the theorem 2.1 finally embeds those integrable equations in ME. In a sense this justyfies the name *master equation* for the equation (1).

The way in which the theorem 2.1 has been proved suggests a method to obtain analytic solutions of $sl((N, \boldsymbol{C})$ SDYM equation (2.8) from some solutions of ME. Given a solution $\Theta(t, \hbar, y, \tilde{y}, z, \tilde{z}, x^1, x^2)$ of (2.12) analytic in all coordinates and in t and $\hbar$ as well for which the Cauchy data read

$$\begin{aligned} \Theta|_{y=0} &= \sum_{\vec{m}=-\vec{M}}^{\vec{M}} \Theta^{(0)}_{\vec{m}} (t, \hbar, \tilde{y}, z, \tilde{z}) E_{\vec{m}} \\ \partial_y \Theta|_{y=0} &= \sum_{\vec{m}=-\vec{M}}^{\vec{M}} \Theta^{(1)}_{\vec{\mu}} (t, \hbar, \tilde{y}, z, \tilde{z}) E_{\vec{m}} \end{aligned} \tag{2.16}$$

by applaying the homomorphism χ_N we get

$$\begin{aligned} \chi_N(\Theta|_{y=0}) &= \sum_{\vec{m}=-\vec{M}}^{\vec{M}} \Theta^{(0)}_{\vec{m}} (t, \frac{2\pi}{N}, \tilde{y}, z, \tilde{z}) L_{\vec{m}} = \sum_{\vec{\mu}} \vartheta^{(0)}_{\vec{\mu}} (t, \tilde{y}, z, \tilde{z}) L_{\vec{\mu}} \\ \chi_N(\partial_y \Theta|_{y=0}) &= \end{aligned} \tag{2.17}$$

$$= \sum_{\vec{m}=-\vec{M}}^{\vec{M}} \Theta^{(1)}_{\vec{\mu}}(t, \frac{2\pi}{N}, \tilde{y}, z, \tilde{z}) L_{\vec{m}} = \sum_{\vec{\mu}} \vartheta^{(1)}_{\vec{\mu}}(t, \tilde{y}, z, \tilde{z}) L_{\vec{\mu}}$$

where the index $\vec{\mu}$ runs through $(0,1), (1,0), ..., (N-1, N-1)$ according to the remark 3.

If one considers (2.17) to be the Cauchy data for $sl((N, \boldsymbol{C})$ SDYM equation written in the form (2.10) then one gets a respective solution ϑ of this equation. One easily finds that the ϑ can be obtained directly from Θ as follows:
One expands Θ into Fourier series in the basis $E_{\vec{m}}$

$$\Theta = \sum_{\vec{m} \in \boldsymbol{Z} \times \boldsymbol{Z}} \Theta_{\vec{m}}(t, \hbar, w, z, \tilde{w}, \tilde{z}) E_{\vec{m}}.$$

Then one makes the substitutions $\hbar \mapsto \frac{2\pi}{N}$ and $E_{\vec{m}} \mapsto L_{\vec{m}}$. At this stage a formal series in $L_{\vec{m}}$ is obtained, for which one applies the property 1 of the proposition 2.1 and gathers the elements standing at the same vectors $L_{\vec{\mu}}$
Consequently, one gets

$$\vartheta = \sum_{\vec{\mu}} (\sum_{\vec{r} \in \boldsymbol{Z} \times \boldsymbol{Z}} (-1)^{(\mu_1+1)r_2 + (\mu_2+1)r_1 + N r_1 r_2} \Theta_{\vec{\mu}+N\vec{r}}) L_{\vec{\mu}}. \tag{2.18}$$

The above procedure will be used in the example 2 to construct a sequence of $su(N)$ chiral fields defined by the solution Θ of ME given in the example 1.

3 From chiral fields to heavenly spaces

Once again consider the reduced form (1.4) of ME discussed in Section 1. Let the symplectic manifold be the same as in section 2, i.e. the two dimensional torus T^2 with coordinates $x^1 \equiv p$, $x^2 \equiv q$ and symplectic form $\boldsymbol{\omega} = dp \wedge dq$. In the same way as is done for the SDYM equation (theorem 2.1) the homomorphism χ_{N} projects the solution Θ of (1.4) onto the solution $\vartheta = \chi_{\text{N}}(\Theta)$: $\mathcal{M}^2 \to sl(N, \boldsymbol{C})$, $N = 2, 3, ...$ of the following equation

$$\partial_w^2 \vartheta + \partial_z^2 \vartheta + [\partial_w \vartheta\, ,\, \partial_z \vartheta] = 0. \tag{3.1}$$

This equation describes the chiral field, i.e. harmonic map $g : \mathcal{M}^2 \to SL(N, \boldsymbol{C})$.

Indeed, each harmonic map on two dimensional manifold $\mathcal{M}^2$ with coordinates $\{w, z\}$ and diagonal metric $\eta_{\alpha\beta} = \text{diag}(+,+)$ fulfils the chiral equation

$$\partial_w(g^{-1}\partial_w g) + \partial_z(g^{-1}\partial_z g) = 0. \tag{3.2}$$

If one denotes $A_\alpha = g^{-1}\partial_\alpha g$, $\alpha \in \{w, z\}$, then this substitution implies the integrability condition $\partial_w A_z - \partial_z A_w + [A_w, A_z] = 0$. Thus the chiral equation takes the form of the system

$$\begin{aligned} &\partial_w A_z - \partial_z A_w + [A_w, A_z] = 0 \\ &\partial_w A_w + \partial_z A_z = 0. \end{aligned} \tag{3.3}$$

From the second equations of (3.3) one infers that there exists $\vartheta : \mathcal{M}^2 \to sl(N, \boldsymbol{C})$ such that $A_w = -\partial_z \vartheta$ and $A_z = \partial_w \vartheta$. Substituting these A_w and A_z into the first equation of (3.3) we get (3.1).

In the present case theorem 2.1 of the previous section asserts that given a suitable solution Θ for $\hbar = \frac{2\pi}{N}$ one obtains the sequence of chiral fields $\vartheta = \chi_{\rm N}(\Theta)$, $N = 2, 3, ...$ satysfying (3.1). On the other hand, it is known that the solution Θ tends in a classical limit $\hbar \to 0$ to the solution θ of the HP heavenly equation (see section 1). This makes the following diagram commutative

$$\begin{tikzcd}
& \Theta \arrow[dl, "\chi_{\rm N}"'] \arrow[dr, "\hbar \to 0"] & \\
\vartheta \arrow[rr, "N \to \infty"'] & & \theta
\end{tikzcd}$$

From this diagram one can conclude that the limiting process $\hbar \to 0$ is equivalent to $N \to \infty$. It suggests that a better understanding of the $sl(\infty, \boldsymbol{C})$ algebra (see Ward 1992) can be achieved by lifting the Poisson algebra to the Moyal bracket algebra.

Remark 4

- The construction presented above allows one to interpret the heavenly equation as a chiral field equation on $\mathcal{M}^2$ with value in the algebra of hamiltonian vector fields on T^2. This algebra corresponds to the group of diffeomorphisms $SDiff(T^2)$ which leave the volume form $\boldsymbol{\omega}$ unchanged.
- This construction gives also a positive answer to the question asked by Ward (1990b): *Can one construct a sequence of $SU(N)$ chiral fields, for $N = 2, 3, ...$, tending to a curved space in the limit?*

In what follows we give an example of sequence of $su(N)$ chiral fields tending to a curved heavenly space for $N \to \infty$. (First, an example of sequence of $sl(N, \boldsymbol{C})$ chiral fields definig a heavenly space in a limit was found in (Przanowski et al 1998)).

Example 2

In the example 1 we have found the solution Θ of the equation (1.4)

$$\Theta = \frac{\pi}{2}\cos(p+q) - w\sin q - \int_0^z \sin(\zeta \frac{2}{\hbar}\sin\frac{\hbar}{2} \cdot \cos q + p) d\zeta. \qquad (3.4)$$

The Θ is analytic in $\hbar$, which is the necessary condition for existence of $\chi_{\rm N}$. It describes in a classical limit a heavenly metric (1.9).

In order to construct a sequence of $su(N)$ chiral fields we assume that the coordinates w, z are real. The Cauchy data for (3.4) read

$$\begin{aligned} \Theta|_{z=0} &= \frac{\pi}{2}\cos(p+q) - w\sin q \\ \partial_z\Theta|_{z=0} &= -\sin p \end{aligned}$$

As has been described in section 2 these Cauchy data induce the Cauchy data for the $su(N)$ chiral field equation (3.1)

$$\begin{aligned} \vartheta|_{z=0} &= \frac{\pi}{2}\,\frac{1}{2}[L_{(1,1)} + (-1)^N L_{(N-1,N-1)}] - w\,\frac{1}{2i}[L_{(0,1)} + L_{(0,N-1)}] \\ \partial_z\vartheta|_{z=0} &= -\frac{1}{2i}[L_{(1,0)} + L_{(N-1,0)}] \end{aligned}$$

as

$$\begin{aligned} \chi_{\rm N}(\cos(p+q)) &= \frac{1}{2}[L_{(1,1)} + L_{(-1,-1)}] = \frac{1}{2}[L_{(1,1)} + (-1)^N L_{(N-1,N-1)}] \\ \chi_{\rm N}(\sin q) &= \frac{1}{2i}[L_{(0,1)} - L_{(0,-1)}] = \frac{1}{2i}[L_{(0,1)} + L_{(0,N-1)}] \\ \chi_{\rm N}(\sin p) &= \frac{1}{2i}[L_{(1,0)} - L_{(-1,0)}] = \frac{1}{2i}[L_{(1,0)} + L_{(N-1,0)}] \end{aligned}$$

The expansion of Θ into Fourier series reads

$$\begin{aligned} \Theta = {}& \frac{\pi}{4}[e^{i(p+q)} + e^{-i(p+q)}] - w\frac{1}{2i}[e^{iq} - e^{-iq}] + \\ & + \sum_{m=1}^{\infty} A_{2m-1}\,\frac{1}{2}[e^{i(p+(2m-1)q)} + e^{-i(p+(2m-1)q)} + e^{i(-p+(2m-1)q)} + e^{-i(-p+(2m-1)q)}] \\ & + \sum_{m=1}^{\infty} A_{2m}\,\frac{1}{2i}[e^{i(p+2mq)} - e^{-i(p+2mq)} - e^{i(-p+2mq)} + e^{-i(-p+2mq)}] + A_0\,\frac{1}{2i}[e^{ip} - e^{-ip}] \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} A_{2m-1} &= \frac{(-1)^m}{\frac{2}{\hbar}\sin\frac{\hbar}{2}}\int_0^{z\frac{2}{\hbar}\sin\frac{\hbar}{2}} J_{2m-1}(\zeta)d\zeta \qquad m = 1, 2, ... \\ A_{2m} &= \frac{(-1)^{m+1}}{\frac{2}{\hbar}\sin\frac{\hbar}{2}}\int_0^{z\frac{2}{\hbar}\sin\frac{\hbar}{2}} J_{2m}(\zeta)d\zeta \qquad m = 0, 1, 2, ... \end{aligned}$$

and J_ℓ is the Bessel function of rank ℓ.

The formal series of $sl(N, \boldsymbol{C})$ matrices reads

$$\begin{aligned} \vartheta = {}& \frac{\pi}{4}[L_{(1,1)} + L_{(-1,-1)}] - w\frac{1}{2i}[L_{(0,1)} - L_{(0,-1)}] + \\ & + \sum_{m=1}^{\infty} A_{2m-1}\,\frac{1}{2}[L_{(1,2m-1)} + L_{(-1,-(2m-1))} + L_{(-1,2m-1)} + L_{(1,-(2m-1))}] \\ & + \sum_{m=1}^{\infty} A_{2m}\,\frac{1}{2i}[L_{(1,2m)} - L_{(-1,-2m)} - L_{(-1,2m)} + L_{(1,-2m)}] + A_0\,\frac{1}{2i}[L_{(1,0)} - L_{(-1,0)}] \end{aligned} \tag{3.6}$$

The cases of N even or odd are considered separetly.
Case 1. N is even.

From proposition 2.1 one has

- $L_{(-1,-1)} = (-1)^{(N-1+1)(-1)+(N-1+1)(-1)+N(-1)(-1)} L_{(N-1,N-1)} = L_{(N-1,N-1)}$
- $L_{(0,-1)} = -L_{(0,N-1)}$
- $L_{(-1,0)} = -L_{(N-1,0)}$
- $L_{(1,2m-1)} = L_{(1,\mu)+(0,Nr)} = (-1)^{(1+1)r} L_{(1,\mu)} = L_{(1,\mu)}$
  where $\mu + Nr = 2m-1$ and $0 \le \mu \le N-1$. Thus $\mu = 2(m - \frac{N}{2}r) - 1$, if one denotes $\mu = 2\nu - 1$ then $m = \nu + \frac{N}{2}r$ and the sum $\sum_{m=1}^{\infty} \mapsto \sum_{\nu=1}^{\frac{N}{2}} \sum_{r=0}^{\infty}$
- $L_{(-1,2m-1)} = L_{(N-1,\mu)}$ and $m = \nu + \frac{N}{2}r$ as above
- $L_{(1,-(2m-1))} = L_{(1,N-\mu)}$
- $L_{(-1,-(2m-1))} = L_{(N-1,N-\mu)}$
- $L_{(1,2m)} = L_{(1,\mu+Nr)} = L_{(1,\mu)}$ where $\mu = 2\sigma$ and $\sigma = 0, 1, ..., \frac{N}{2} - 1$
- $L_{(-1,2m)} = -L_{(N-1,2m)} = -L_{(N-1,\mu)}$ where $\mu = 2\sigma$ and $\sigma = 0, 1, ..., \frac{N}{2} - 1$
- $L_{(1,-2m)} = L_{(1,-\mu-Nr)} = L_{(1,N-\mu)}$ where $\mu = 2\sigma$ and $\sigma = 0, 1, ..., \frac{N}{2} - 1$
- $L_{(-1,-2m)} = -L_{(N-1,N-\mu)}$ where $\mu = 2\sigma$ and $\sigma = 0, 1, ..., \frac{N}{2} - 1$

Substituting all that into (3.6) one gets

$$\begin{aligned} \vartheta &= \frac{\pi}{2} \underbrace{\frac{1}{2}(L_{(1,1)} + L_{(N-1,N-1)})} - w \underbrace{\frac{1}{2i}(L_{(0,1)} + L_{(0,N-1)})} + \\ &\quad + \sum_{\nu=1}^{\frac{N}{2}} a_{2\nu-1} [\underbrace{\frac{1}{2}(L_{(1,2\nu-1)} + L_{(N-1,N-(2\nu-1))})} + \underbrace{\frac{1}{2}(L_{(N-1,2\nu-1)} + L_{1,N-(2\nu-1)})}] \\ &\quad + \sum_{\nu=1}^{\frac{N}{2}-1} a_{2\nu} [\underbrace{\frac{1}{2i}(L_{(1,2\nu)} + L_{(N-1,N-2\nu)})} + \underbrace{\frac{1}{2i}(L_{(N-1,2\nu)} + L_{(1,N-2\nu)})}] \\ &\quad + a_0 \underbrace{\frac{1}{2i}(L_{(1,0)} + L_{(N-1,0)})} \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} a_{2\nu-1} &= \frac{(-1)^{\nu}}{\frac{N}{\pi}\sin\frac{\pi}{N}} \sum_{r=0}^{\infty} (-1)^{\frac{N}{2}r} \int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{2\nu-1+Nr}(\zeta) d\zeta, \qquad \nu = 1, ..., \frac{N}{2} \\ a_{2\nu} &= \frac{(-1)^{\nu+1}}{\frac{N}{\pi}\sin\frac{\pi}{N}} \sum_{r=0}^{\infty} (-1)^{\frac{N}{2}r} \int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{2\nu+Nr}(\zeta) d\zeta, \qquad \nu = 1, ..., \frac{N}{2} - 1 \\ a_0 &= \frac{-1}{\frac{N}{\pi}\sin\frac{\pi}{N}} \left(2\sum_{r=0}^{\infty} (-1)^{\frac{N}{2}r} \int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{Nr}(\zeta) d\zeta + \int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_0(\zeta) d\zeta \right) \end{aligned}$$

Case 2. N is odd.

In this case

- $L_{(-1,-1)} = (-1)^{(N-1+1)(-1)+(N-1+1)(-1)+N(-1)(-1)} L_{(N-1,N-1)} = -L_{(N-1,N-1)}$
- $L_{(0,-1)} = -L_{(0,N-1)}$
- $L_{(-1,0)} = -L_{(N-1,0)}$
- $L_{(1,2m-1)} = L_{(1,\mu)+(0,Nr)} = (-1)^{(1+1)r} L_{(1,\mu)} = L_{(1,\mu)}$
  where $\mu + Nr = 2m-1$ and $0 \leq \mu \leq N-1$.

Observe that if μ is odd then r must be even and vice versa if μ is even then r must be odd. We divide the sum $\sum_{m=1}^{\infty} \mapsto \sum_{\mu=1}^{N-1} \sum_{r=0}^{\infty}$ into two separete sums

$$\left\{ \begin{array}{ll} \mu = 2\nu - 1 & \nu = 1, ..., \frac{N-1}{2} \\ r = 2k & k = 0, 1, ..., \infty \\ m = \nu + Nk & \end{array} \right. \qquad \left\{ \begin{array}{ll} \mu = 2\nu & \nu = 0, 1, ..., \frac{N-1}{2} \\ r = 2k+1 & k = 0, 1, ..., \infty \\ m = \nu + \frac{N+1}{2} + Nk & \end{array} \right.$$

- $L_{(-1,2m-1)} = (-1)^r L_{(N-1,\mu)}$ and $m = \nu + \frac{N}{2} r$ as above
- $L_{(1,-(2m-1))} = L_{(1,N-\mu)}$
- $L_{(-1,-(2m-1))} = (-1)^{r+1} L_{(N-1,N-\mu)}$
- $L_{(1,2m)} = L_{(1,\mu+Nr)} = L_{(1,\mu)}$

where this time $2m = \mu + Nr$ and μ odd implies r odd and μ even implies r even.

$$\left\{ \begin{array}{ll} \mu = 2\sigma - 1 & \sigma = 1, ..., \frac{N-1}{2} \\ r = 2l+1 & l = 0, 1, ..., \infty \\ m = \sigma + \frac{N-1}{2} + Nl & \end{array} \right. \qquad \left\{ \begin{array}{ll} \mu = 2\sigma & \sigma = 1, ..., \frac{N-1}{2} \\ r = 2l & l = 0, 1, ..., \infty \quad \text{if} \\ (\ \sigma = 0 & \Rightarrow \quad l \neq 0\ ) \\ m = \sigma + Nl & \end{array} \right.$$

- $L_{(-1,2m)} = -L_{(N-1,2m)} = (-1)^{r+1} L_{(N-1,\mu)}$
- $L_{(1,-2m)} = L_{(1,-\mu-Nr)} = L_{(1,N-\mu)}$
- $L_{(-1,-2m)} = (-1)^r L_{(N-1,N-\mu)}$

Substituting all that into (3.6) we obtain

$$\begin{aligned} \vartheta \quad = \quad & \frac{\pi}{2} \underbrace{\frac{1}{2}(L_{(1,1)} - L_{(N-1,N-1)})} - w \underbrace{\frac{1}{2i}(L_{(0,1)} + L_{(0,N-1)})} + \\ & + \sum_{\nu=1}^{\frac{N-1}{2}} a_{2\nu-1} [\underbrace{\frac{1}{2}(L_{(1,2\nu-1)} - L_{(N-1,N-(2\nu-1))})} + \underbrace{\frac{1}{2}(L_{(N-1,2\nu-1)} + L_{(1,N-(2\nu-1))})}] \end{aligned} \tag{3.8}$$

$$+\sum_{\nu=1}^{\frac{N-1}{2}} a_{2\nu}[\underbrace{\frac{1}{2}(L_{(1,2\nu)}+L_{(N-1,N-2\nu)})}+\underbrace{\frac{1}{2}(L_{(1,N-2\nu)}-L_{(N-1,2\nu)})}]$$

$$+\sum_{\nu=1}^{\frac{N-1}{2}} b_{2\nu-1}[\underbrace{\frac{1}{2i}(L_{(1,2\nu-1)}+L_{(N-1,N-(2\nu-1))})}+\underbrace{\frac{1}{2i}(L_{(1,N-(2\nu-1))}-L_{(N-1,2\nu-1)})}]$$

$$+\sum_{\nu=1}^{\frac{N-1}{2}} b_{2\nu}[\underbrace{\frac{1}{2i}(L_{(1,2\nu)}-L_{(N-1,N-2\nu)})}+\underbrace{\frac{1}{2i}(L_{(N-1,2\nu)}+L_{(1,N-2\nu)})}]$$

$$+2a_0\underbrace{\frac{1}{2}[L_{(1,0)}-L_{(N-1,0)}]}+b_0\underbrace{\frac{1}{2i}[L_{(1,0)}+L_{(N-1,0)}]}$$

where

$$\begin{aligned}
a_{2\nu-1} &= \frac{(-1)^{\nu}}{\frac{N}{\pi}\sin\frac{\pi}{N}}\sum_{k=0}^{\infty}(-1)^k\int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{2\nu-1+N2k}(\zeta)d\zeta \qquad \nu=1,2,...,\frac{N-1}{2}\\
a_{2\nu} &= \frac{(-1)^{\nu+\frac{N+1}{2}}}{\frac{N}{\pi}\sin\frac{\pi}{N}}\sum_{k=0}^{\infty}(-1)^k\int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{2\nu+N(2k+1)}(\zeta)d\zeta \qquad \nu=0,1,2,...,\frac{N-1}{2}\\
b_{2\nu-1} &= \frac{(-1)^{\nu+\frac{N+1}{2}}}{\frac{N}{\pi}\sin\frac{\pi}{N}}\sum_{k=0}^{\infty}(-1)^k\int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{2\nu-1+N(2k+1)}(\zeta)d\zeta \qquad \nu=1,2,...,\frac{N-1}{2}\\
b_{2\nu} &= \frac{(-1)^{\nu+1}}{\frac{N}{\pi}\sin\frac{\pi}{N}}\sum_{k=0}^{\infty}(-1)^k\int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{2\nu+N2k}(\zeta)d\zeta \qquad \nu=1,2,...,\frac{N-1}{2}\\
b_0 &= -\frac{1}{\frac{N}{\pi}\sin\frac{\pi}{N}}[2\sum_{k=1}^{\infty}(-1)^k\int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_{2Nk}(\zeta)d\zeta+\int_0^{z\frac{N}{\pi}\sin\frac{\pi}{N}} J_0(\zeta)d\zeta]
\end{aligned}$$

In the formulas (3.7) and (3.8) the linear combinations denoted by underbraces are elements of the basis of $su(N)$ for N even or odd, respectively.

Finally (3.7) and (3.8) give the sequence of $su(N)$ chiral fields tending for $N\to\infty$ to the heavenly space described by the metric (1.9).

In particular, substituting $N=2$ into (3.7) and applying the formulas

$$\begin{aligned}
\sum_{k=1}^{\infty}(-1)^k J_{2k+1}(\zeta) &= \frac{\sin\zeta}{\zeta}\\
2\sum_{k=1}^{\infty}(-1)^k J_{2k}(\zeta)+J_0(\zeta) &= \cos\zeta
\end{aligned}$$

and the fact that

$$L_{(1,1)}=-\frac{i}{\pi}\sigma_1 \ , \quad \frac{1}{i}L_{(0,1)}=\frac{i}{\pi}\sigma_2 \ , \quad \frac{1}{i}L_{(1,0)}=\frac{i}{\pi}\sigma_3$$

where $\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices one gets

$$\vartheta=\frac{1}{2i}\cos(\frac{2}{\pi}z)\sigma_1+\frac{w}{\pi i}\sigma_2+\frac{1}{2i}\sin(\frac{2}{\pi}z)\sigma_3. \qquad (3.9)$$

Conclusions

The purpose of this paper was to justify the name *master equation* (ME) for equation (1). In particular we were able to show (in theorem 2.1) that any analytic solution of the $sl(N, \boldsymbol{C})$ equation could be obtained from a suitable solution of ME. This embedding holds true also for the solutions of the HP heavenly equation as well as for the solutions of the $sl(N, \boldsymbol{C})$ chiral equation in 2 diemensions or Ward's integrable chiral equations in $2+1$ diemensions (Ward 1988, Ward 1995, Przanowski and Formański 1999, Dunajski and Manton 2005).

Moreover, as has been shown in section 3 there exists a natural method to construct sequences of su(N) (or $sl(N, \boldsymbol{C})$) chiral fields tending to heavenly spaces when $N \to \infty$. It would be very interesting to apply the described procedure to the finite energy solutions of chiral field equation (*unitons* ; Uhlenbeck (1989), Ward (1990a)) or Ward's integrable chiral equations in 2+1 diemensions. As a result one would obtain a heavenly space. The question arises wheather this space is one of a finite action (an *instanton*).

Another interesting open problem is to try to embed the *Kadomtsev-Petviashvili* (KP) equation into ME. This posibility is suggested by (Mason 1990, Mason and Woodhouse 1996, Strachan 1997, Przanowski and Formański 1999). We expect that the extension of the Poisson algebra to its quantum deformation is sufficient to encode KP equation in ME.

Of cource the main technical problem is to find an effective method of looking for solutions of ME.

Acknowledgements

We are indebted to Maciej Dunajski and Jacek Tafel for the interest in this work and a very fruitful comments. The work was partially supported by CONACyT grant no: 41993-F (Mexico) and NATO grant no: PST.CLG.978984

References

[1] Chakravarty S and Ablowitz M J 1992, *On reductions of self-dual Yang-Mills equations*, in: *Painlevé Transcendents* eds. D. Levi, P. Winternitz, (Plenum Press, New York) 331-343.

[2] De Wilde M and Lecomte P B A 1983, *Lett. Math. Phys.* **7**, 487-96 .

[3] Dunajski M and Manton N S 2005, *Nonlinearity* **18**, 1677-89.

[4] Husain V 1994, *Class. Quantum Grav.* **11**, 927-37.

[5] Fairlie D B, Fletcher P and Zachos C K 1990, *J. Math. Phys.* **31** (5), 1088-94.

[6] Fedosov B V 1994, *J. Diff. Geom.* **40**, 213-38.

[7] Fedosov B V 1996, *Deformation Quantization and Index Theory* (Akademie Verlag, Berlin).

[8] Formański S 2004, *Moyal deformation of heavenly equations. Integrability and relations to other non-linear equations*. PhD thesis. Tech. Univ. of Lodz (in Polish)

[9] Formański S and Przanowski M 2005, *J. Phys. A: Math. Gen.* **38**, 4399-418.

[10] Leznov A N 1988, *Theor. M. Phys.* **73**, 1233-37

[11] Mason L J 1990, *Twistor Newsletter*, **30**, 14-17.

[12] Mason L J and Sparling G A J 1989, *Phys. Lett.* **A 137**, 29-33.

[13] Mason L J and Woodhouse N M J 1996, *Integrability, self-duality, and twistir theory* (Clarendon Press, Oxford).

[14] E.T. Newman E T 1978, *Phys. Rev.* **D18**, 2901-8.

[15] Parkes A 1992, *Phys. Lett.* **B 286**, 265-70.

[16] Park Q-H 1992, *Int. J. Mod. Phys.* **A 7**, No. 7, 1415-47.

[17] J.F. Plebański J F 1975, *J. Math. Phys.* **16** 2395-402.

[18] Plebański J F and Przanowski M 1988, *Acta Phys. Pol.* **B19**, 805-29.

[19] Przanowski M, Formański S and Turrubiates F J 1998, *Mod. Phys. Lett.* **A 13**, No. 40, 3193-200.

[20] Przanowski M and Formański S 1999, *Acta Phys. Pol.* **B 30** , 863-79.

[21] Strachan I A B 1997, *J. Geom. Phys.* **21**, 255-78.

[22] Tafel J 1993, *J. Math. Phys.* **34** 1892-907.

[23] Uhlenbeck K 1989, *J. Diff. Geom.* **30**, 1-50.

[24] Ward R S 1985, *Philos. Trans. R.Soc.* **A 315** 451-57.
R.S. Ward, *Multi-dimensional integrable systems* ,in: *Field theory, quantum gravity and strings,* eds. H.J. de Vega and N. Sanchez, Lecture Notes in Physics **280** (Springer, Berlin 1986).
R.S. Ward, *Integrable systems in twistor theory,* in: *Twistors in mathematics and physics,* eds. T.N. Bailey and R.J. Baston

[25] Ward R S 1990a, *Comm. Math. Phys.* **128**, 319-32.

[26] Ward R S 1990b, *Class. Quantum Grav.* **7**, L217-22.

[27] Ward R S 1992, *J. Geom. Phys.* **8**, 317-25.